

Norm controlled inversions and a corona theorem for H^∞ -quotient algebras

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Abstract

Let Θ be an inner function on the unit disc \mathbb{D} . We give a description of those Θ for which the quotient algebra $H^\infty/\Theta H^\infty$ has no corona with respect to the visible part of its spectrum, that is for which $\mathfrak{M}(H^\infty/\Theta H^\infty) = \{z \in \mathbb{D}: \Theta(z) = 0\} \mathfrak{M}(H^\infty)$. It happens that this property is equivalent to the norm controlled inversion property for $H^\infty/\Theta H^\infty$, as well as to a kind of weakened Carleson type embedding theorem. The quotient algebra $A(\mathbb{D})/\Theta H^\infty$ is also considered. An interpretation of our main results in terms of model operators is given, too.

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1. Introduction

The problem we are interested in here is that of efficient inversion, as it is treated in [1,11], and subsequent papers. This problem is described in detail below.

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Let A be a commutative unital Banach algebra with unit e and $\mathfrak{M}(A)$ its maximal ideal space. For $a \in A$, let \hat{a} denote the Gelfand transform of a . We let

$$\delta(a) = \min_{t \in \mathfrak{M}(A)} |\hat{a}(t)|.$$

Note that $\delta(a) \leq \|\hat{a}\|_\infty \leq \|a\|$. When $a = (a_1, \dots, a_n) \in A^n$ we define

$$\delta_n(a) = \min_{t \in \mathfrak{M}(A)} |\hat{a}(t)|,$$

where $|\hat{a}(t)| = (\sum_{j=1}^n |\hat{a}_j(t)|^2)^{1/2}$ for $t \in \mathfrak{M}(A)$ and we let

$$\|a\| = \left(\sum_{j=1}^n \|a_j\|^2 \right)^{1/2}.$$

Let δ be a real number satisfying $0 < \delta \leq 1$. We are interested in finding, or bounding, the functions

$$c_1(\delta) = \sup\{\|a^{-1}\| : \|a\| \leq 1, \delta(a) \geq \delta\}$$

and

$$c_n(\delta) = \sup\left\{\inf\left\{\|b\| : \sum_{j=1}^n a_j b_j = e\right\}, \|a\| \leq 1, \delta_n(a) \geq \delta\right\}.$$

In fact, often only a part of $\mathfrak{M}(A)$ is available, say $\Lambda \subset \mathfrak{M}(A)$. We call this a *visible* part of $\mathfrak{M}(A)$. (We will present examples below.) In this case, we would like to bound $\|a^{-1}\|$ in terms of the visible part of the spectral data. Thus, we define the counterpart of $\delta(a)$ as

$$\delta(a, \Lambda) = \inf_{t \in \Lambda} |\hat{a}(t)|.$$

In a similar way, we consider $\delta_n(a, \Lambda)$. This, in turn, yields modified functions $c_n(\Lambda, \delta)$ for $n \geq 1$ (if it is not readily apparent which algebra we mean, we write $c_n(A, \Lambda, \delta)$). If a is not invertible, we define $\|a^{-1}\| = \infty$. If $\delta_n(a, \Lambda) \geq \delta$ and $\|a\| \leq 1$, we let $\|b\| = \infty$, if the set of all $b = (b_1, \dots, b_n)$ for which $\sum_{j=1}^n a_j b_j = e$ is empty. It should be clear that $1 \leq c_n(\Lambda, \delta) \leq c_{n+1}(\Lambda, \delta)$ and, if $0 < \delta' \leq \delta \leq 1$, then $c_n(\Lambda, \delta) \leq c_n(\Lambda, \delta')$. This implies the existence of a *critical constant*, denoted here by $\delta_n(A, \Lambda)$ (or simply $\delta_n(A)$ if $\Lambda = \mathfrak{M}(A)$) such that

$$c_n(\Lambda, \delta) = \infty \quad \text{for } 0 < \delta < \delta_n(A, \Lambda) \quad \text{and} \quad c_n(\Lambda, \delta) < \infty \quad \text{for } \delta_n(A, \Lambda) < \delta \leq 1.$$

If A is a uniform algebra, then $\delta_1(A) = 0$.

For elements a with $0 < \delta_n(a, \Lambda) < \delta_n(A, \Lambda)$ we can say that the inversion problem is *ill posed* in the sense that there is no control of inverses in terms of visible spectral data. For elements with $\delta_n(A, \Lambda) < \delta_n(a, \Lambda) \leq 1$ the problem is said to be well posed; that is, there is such an estimate.

Let H^∞ be the algebra of bounded holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ endowed with the supremum norm $\|f\|_\infty = \sup_{|z| < 1} |f(z)|$ and let $A(\mathbb{D}) = H^\infty \cap C(\overline{\mathbb{D}})$ be the disc algebra. In this paper, given an inner function Θ , we characterize those quotient algebras $A = H^\infty/\Theta H^\infty$ or $A = A(\mathbb{D})/\Theta H^\infty$ for which the inversion problem is well posed; that is for which $\delta_1(A, \Lambda) = 0$.

Note that we interpret $A(\mathbb{D})/\Theta H^\infty$ to mean the canonical image of $A(\mathbb{D})$ in the quotient algebra $H^\infty/\Theta H^\infty$. It is known that $A(\mathbb{D})/\Theta H^\infty$ is a closed subalgebra of $H^\infty/\Theta H^\infty$ if and only if $A(\mathbb{D}) + \Theta H^\infty$ is norm closed in H^∞ . It is also an interesting fact [8,17] that the sum $C(\mathbb{T}) + \Theta H^\infty$ is closed in $L^\infty(\mathbb{T})$ if and only if either $m(\sigma(\Theta) \cap \mathbb{T}) = 0$ or $m(\sigma(\Theta) \cap \mathbb{T}) = 1$, where m denotes normalized Lebesgue measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$.

Having an inner function Θ and a function $f \in H^\infty$, we define the *visible spectrum* of an element $f + \Theta H^\infty$ in $A = H^\infty/\Theta H^\infty$ as the range

$$f(\Lambda) = \{f(\lambda): \lambda \in \Lambda\},$$

where $\Lambda = \sigma(\Theta) \cap \mathbb{D} = \{z \in \mathbb{D}: \Theta(z) = 0\}$, $\sigma(\Theta)$ being the spectrum of Θ (see below).

We show (Theorems 3.3 and 3.4) that $\delta_1(A, \Lambda) = 0$ if and only if $\delta_n(A, \Lambda) = 0$ for every $n \geq 1$, and this happens if and only if Λ is dense in the maximal ideal space $\mathfrak{M}(A)$.

We also characterize the latter property in terms of the canonical factorization $\Theta = S_\mu B$, where

$$B(z) = \prod_{j \geq 1} b_{\lambda_j}(z), \quad b_\lambda(z) = \frac{|\lambda|}{\lambda} \cdot \frac{\lambda - z}{1 - \bar{\lambda}z}$$

is the Blaschke product with the zeros of Θ (repeated according to their multiplicities), and

$$S_\mu(z) = \exp\left(\int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} d\mu(\zeta)\right)$$

stands for the corresponding singular inner function (μ is a measure on the circle $\mathbb{T} = \{z: |z| = 1\}$, $\mu \geq 0$, singular with respect to Lebesgue measure m). The spectrum of an inner function Θ is defined as

$$\sigma(\Theta) = \text{supp}(\mu) \cup \overline{\{z \in \mathbb{D}: \Theta(z) = 0\}}^{\mathbb{C}}.$$

Recall that, following the standard terminology (see [2,7,10]), a function algebra A on a set Λ has no corona if the set Λ is dense in the maximal ideal space $\mathfrak{M}(A)$. With this notation we show (Theorems 3.3 and 3.4) that the algebra $H^\infty/\Theta H^\infty$, being considered on the set $\Lambda = \sigma(\Theta) \cap \mathbb{D}$ has no corona if and only if the following *weak embedding property* (WEP) holds:

for every $\epsilon > 0$ there exists $\eta > 0$ such that

$$\{z \in \mathbb{D}: |\Theta(z)| < \eta\} \subset \left\{z \in \mathbb{D}: \inf_{\lambda \in \Lambda} |b_\lambda(z)| < \epsilon\right\}.$$

We call this an embedding property because the latter condition is related to the famous Carleson embedding property (CEP) appearing in the Carleson interpolation theorem. Namely, the Carleson embedding means that

$$\sum_{j \geq 1} \frac{(1 - |\lambda_j|^2)(1 - |z|^2)}{|1 - \bar{\lambda}_j z|^2} \leq C$$

for every $z \in \mathbb{D}$ (see [10, p. 151]), whereas for a Blaschke product Θ with zero sequence (λ_j) the WEP is equivalent to

$$\sum_{j \geq 1} \frac{(1 - |\lambda_j|^2)(1 - |z|^2)}{|1 - \bar{\lambda}_j z|^2} \leq C$$

for every $z \in \mathbb{D} \setminus \bigcup_{\lambda \in \Lambda} \{\zeta : |b_\lambda(\zeta)| < \epsilon\}$. It is known that CEP is equivalent to the set Λ being a finite union of Carleson interpolating sequences. Therefore, the condition “ $\Theta = B$ is a finite product of interpolating Blaschke products,” is sufficient for H^∞/BH^∞ to have no corona. A surprising example (Example 3.7, invented in a collaboration with S. Treil and V. Vasyunin) shows that there exist Blaschke products satisfying WEP but not CEP, and so there exist algebras H^∞/BH^∞ without corona (or, equivalently, having the norm controlled inversion property) such that the generating Blaschke product B is not a finite product of interpolating Blaschke products. We mention that, according to Theorems 3.3 and 3.4, the norm controlled inversion property (meaning $\delta_1(H^\infty/\Theta H^\infty) = 0$ and/or the existence of the joint majorant for inverses (solutions of Bezout equations) $c_n(\Lambda, \delta) < \infty$ for all $0 < \delta < 1$) is equivalent to the *simple inverse stability* of the restriction algebra $H^\infty|_\Lambda$:

$$\left(f \in H^\infty, \delta(f, \Lambda) = \inf_{\lambda \in \Lambda} |f(\lambda)| > 0 \right) \Rightarrow (f + \Theta H^\infty \text{ is invertible in } H^\infty/\Theta H^\infty).$$

The latter equivalence fails for the quotient disc algebras $A(\mathbb{D})/\Theta H^\infty$. Indeed, Lemma 4.1 shows that every function $f \in A(\mathbb{D})$ satisfying $\delta(f, \sigma(\Theta)) > 0$ is invertible in $A(\mathbb{D})/\Theta H^\infty$ if and only if $m(\sigma(\Theta) \cap \mathbb{T}) = 0$, whereas the norm controlled inversions (i.e. $\delta_1(A(\mathbb{D})/\Theta H^\infty) = 0$) hold true if and only if $m(\sigma(\Theta) \cap \mathbb{T}) = 0$ and the WEP are satisfied (see Theorem 4.2 below). Moreover, in the case where $m(\sigma(\Theta) \cap \mathbb{T}) = 0$, the maximal ideal space $\mathfrak{M}(A(\mathbb{D})/\Theta H^\infty)$ can be identified with the spectrum $\sigma(\Theta)$. Therefore, as for some other Banach algebras (see examples in [11, 12]), the lack of the norm controlled inversion property for quotient algebras is not necessarily related to the existence of a corona (meaning the existence of a large “invisible spectrum” or difference between the entire maximal ideal space and the “visible part”), but rather a subtle discrepancy between the Gelfand transform norm and the original Banach algebra norm.

The geometric meaning of WEP sequences is still, unfortunately, unclear. Nothing similar to the Carleson density characterization for CEP sequences (see [10, p. 153]) is known. It is worth mentioning a couple of known related properties (for details see the end of Section 3 below): a separated WEP sequence is interpolating and a WEP sequence that is a finite union of separated sequences is a CEP sequence. Recall that a sequence, (λ_j) , is separated if $\inf_{j \neq k} |b_{\lambda_j}(\lambda_k)| > 0$.

Finally, we interpret our results in terms of the spectral properties of the model operators. In fact, the interest in studying just this class of operators was the primary motivation for this paper. Now, let Θ be an inner function and

$$K_{\Theta} = H^2 \ominus \Theta H^2$$

be the model space; that is, the orthogonal complement of the shift-invariant subspace ΘH^2 of the Hardy space

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|^2 = \sum_{k \geq 0} |\hat{f}(k)|^2 < \infty \right\}$$

on the disc \mathbb{D} . The model operator (having Θ as its characteristic function, see [10,13]) is $M_{\Theta} : K_{\Theta} \rightarrow K_{\Theta}$,

$$M_{\Theta} f = P_{\Theta}(zf), \quad f \in K_{\Theta},$$

where P_{Θ} denotes the orthogonal projection on K_{Θ} . It is known that every C_{00} Hilbert space contraction T having unit defect indices, $\text{rank}(I - T^*T) = \text{rank}(I - TT^*) = 1$, is unitarily equivalent to a model operator M_{Θ} . Now we can provide an operator theoretic interpretation of our function theoretic results (see [10,13,18]).

- (i) The spectrum of M_{Θ} coincides with $\sigma(\Theta)$, and the point spectrum (the set of eigenvalues) is $\Lambda = \sigma(\Theta) \cap \mathbb{D}$.
- (ii) The commutant of M_{Θ} , $\{M_{\Theta}\}' = \{A : K_{\Theta} \rightarrow K_{\Theta}, AM_{\Theta} = M_{\Theta}A\}$ coincides with the set of H^{∞} functions of M_{Θ} , and is isometrically isomorphic to the quotient algebra $H^{\infty}/\Theta H^{\infty}$.
- (iii) For $f \in H^{\infty}$, $f(\lambda)$ are eigenvalues of $A = f(M_{\Theta})$, where $\lambda \in \Lambda$.
- (iv) The operator $f(M_{\Theta})$ is invertible if and only if the class $f + \Theta H^{\infty}$ is invertible in the algebra $H^{\infty}/\Theta H^{\infty}$.
- (v) The inner function Θ is a Blaschke product if and only if the eigen- and associated-vectors of M_{Θ} are complete in K_{Θ} .

We are interested in the following questions. Is it possible to guarantee the invertibility of $f(M_{\Theta})$ knowing that the eigenvalues $f(\lambda)$, for $\lambda \in \Lambda$, are bounded away from zero? Is it possible to bound the norm $\|f(M_{\Theta})^{-1}\|$ in terms of the minimum modulus of the eigenvalues $\delta(f, \Lambda) = \inf_{\lambda \in \Lambda} |f(\lambda)| > 0$? Clearly, these questions are equivalent to those answered in Theorems 3.3, 3.4, and 4.2.

The paper is organized as follows. Section 2 contains the main notation and preliminaries. Section 3 is devoted to the equivalence of the norm controlled inversions and the WEP, as well as to a discussion of the latter property. Section 4 treats the case of the algebra $A(\mathbb{D})/\Theta H^{\infty}$.

2. Preliminaries

This section contains the notation and definitions we will need. The main contribution of this section is the newly-developed definition of WEP, or the *Weak Embedding Property*.

Let $b_{\lambda} = \frac{\bar{\lambda}}{|\lambda|} \frac{\lambda - z}{1 - \bar{\lambda}z}$ be a single Blaschke factor and let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the natural numbers. Letting $\kappa : \lambda \rightarrow k_{\lambda}$ be a map from \mathbb{D} to \mathbb{N} satisfying the Blaschke condition $\sum_{\lambda \in \mathbb{D}} k_{\lambda}(1 - |\lambda|) < \infty$ (that is the Blaschke mass of κ is finite) we may define the Blaschke product $B = B(\kappa, \cdot)$ by

$$B(\kappa, z) = \prod_{k_\lambda > 0} b_\lambda(z)^{k_\lambda}.$$

Note that k_λ is just the multiplicity of the zero λ . A singular inner function S is defined by

$$S_\mu(z) = \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu(\xi)\right) \quad \text{for } z \in \mathbb{D},$$

where μ is a nonnegative, finite Borel measure on \mathbb{T} that is singular with respect to Lebesgue measure m . Every inner function Θ on \mathbb{D} can be factored as

$$\Theta = e^{i\theta} B S_\mu,$$

where B is a Blaschke product and S is a singular inner function. We let Λ denote the sequence of zeros of Θ inside \mathbb{D} .

Recall that a Blaschke product B with (simple) zeros (z_n) is said to be an interpolating Blaschke product if $H^\infty|_\Lambda = \ell^\infty(\Lambda)$. The classical Carleson theorem says that B is an interpolating Blaschke product if and only if

$$\delta_C(B) := \inf_n \left| \frac{B}{b_{z_n}}(z_n) \right| = \inf_n (1 - |z_n|^2) |B'(z_n)| > 0.$$

The quantity $\delta_C(B)$ is called the Carleson separation constant for B (or for the corresponding zero-sequence Λ).

If we write $\text{supp}(\kappa) = \{\lambda \in \mathbb{D}: k_\lambda > 0\}$ for the zeros of the Blaschke factor in \mathbb{D} , the spectrum of an inner function Θ is (by definition)

$$\sigma(\Theta) := \overline{\text{supp}(\kappa)} \cup \text{supp}(\mu)$$

(see, for example, [10, p. 62]).

If $\Theta = B$ is a Blaschke product with simple zeros (that is, $k_\lambda \leq 1$ on \mathbb{D}), then $H^\infty/\Theta H^\infty$ is the space of traces of H^∞ on the set Λ ; that is,

$$H^\infty/BH^\infty = H^\infty|_\Lambda$$

endowed with the trace norm given by $\|f\| = \inf\{\|g\|_\infty: g \in H^\infty, g|_\Lambda = f|_\Lambda\}$.

For Blaschke products with higher multiplicities, the algebra H^∞/BH^∞ can be similarly interpreted as a space of germs of height k_λ on Λ .

Let $\rho(z, w) = |(z - w)/(1 - \bar{z}w)|$ denote the pseudohyperbolic distance between two points $z, w \in \mathbb{D}$. For $\lambda \in \mathbb{D}$ and $\epsilon > 0$, let $D_\rho(\lambda, \epsilon)$ denote the pseudohyperbolic disc with center λ and radius ϵ . For a function $\varphi: \mathbb{D} \rightarrow \mathbb{C}$ and $\eta > 0$ we let

$$\Omega_\eta(\varphi) = \{z \in \mathbb{D}: |\varphi(z)| < \eta\}$$

denote the η -level set of φ . The following concept will be the major feature in what follows.

Definition. Let $\Theta = BS_\mu$ be an inner function and let Λ be the zero sequence of Θ in \mathbb{D} . We say that Θ satisfies the Weak Embedding Property (WEP) [for short: $\Theta \in WEP$], if for every $\epsilon > 0$ there exists $\eta > 0$ such that

$$\Omega_\eta(\Theta) \subseteq \bigcup_{\lambda \in \Lambda} D_\rho(\lambda, \epsilon).$$

For a WEP inner function we define

$$\eta_\Theta(\epsilon) = \sup \left\{ \eta > 0: \Omega_\eta(\Theta) \subseteq \bigcup_{\lambda \in \Lambda} D_\rho(\lambda, \epsilon) \right\},$$

and we call η_Θ the WEP characteristic of Θ . For a Blaschke product $B = B_\Lambda$ associated with a Blaschke sequence Λ , we also write $\eta_B = \eta_\Lambda$.

We note that

$$\bigcup_{\lambda \in \Lambda} D_\rho(\lambda, \epsilon) = \Omega_\epsilon \left(\inf_{\lambda \in \Lambda} |b_\lambda| \right).$$

We also note that every WEP inner function has a nontrivial Blaschke factor and the union in our definition above does not depend on the multiplicities of the zeros of the Blaschke factor.

The motivation as to why this property refers to a “weak embedding” is given in the introduction. The WEP inner functions are studied in Sections 3, 4 of this paper. However, we mention here four basic properties of WEP inner functions and WEP characteristics.

(P1) If $\inf_{\lambda \in \Lambda} |b_\lambda(z)| \geq \epsilon > 0$, then $|\Theta(z)| \geq \eta_\Theta(\epsilon)$.

(P2) $\eta_\Theta(\epsilon) \leq \epsilon$ for every $\epsilon > 0$.

Proof. Indeed, since b_λ is a factor of Θ for every $\lambda \in \Lambda$ we see that $|\Theta(z)| \leq |b_\lambda(z)|$ for every $z \in \mathbb{D}$ and $\lambda \in \Lambda$. Therefore, if $\eta_\Theta(\epsilon) > \epsilon$ we would be able to find an η and a z such that $\eta_\Theta(\epsilon) > |\Theta(z)| = \eta > \epsilon$. Applying the definition of $\eta_\Theta(\epsilon)$ we see that for some λ we would have

$$\epsilon < |\Theta(z)| \leq |b_\lambda(z)| < \epsilon,$$

a contradiction. \square

(P3) Let B be a Blaschke product that is the product of N interpolating Blaschke products B_j , $1 \leq j \leq N$. Then B is a WEP Blaschke product and

$$\eta_B(\epsilon) \geq c \cdot \epsilon^N$$

for all ϵ with $0 < \epsilon < 1$ and a convenient constant $c > 0$.

Proof. To see this, we note that it is well known (see, for example [10, p. 218]) that an interpolating Blaschke product B_j associated with the zero set Λ_j is characterized by the lower estimate

$|B_j(z)| \geq c_j \inf_{\lambda \in \Lambda_j} |b_\lambda(z)|$ for every $z \in \mathbb{D}$, where $c_j > 0$ is a convenient constant. Multiplying these inequalities we obtain a constant c such that

$$|B(z)| \geq c \cdot \inf_{\lambda \in \Lambda} |b_\lambda(z)|^N$$

for every $z \in \mathbb{D}$. The result follows. \square

(P4) Let Λ be a sequence in \mathbb{D} satisfying CEP (the definition of which appears in the introduction). Then Λ satisfies WEP with a lower bound for η_Λ , as indicated in (P3) above.

Proof. Indeed, it is well known (see [2,10], or [13]) that a CEP sequence is a finite union of interpolating sequences. The result follows from (P3) above. \square

It will be convenient to have two notations for the zeros of an inner function when dealing with purely topological properties (see (6) below). Thus, we write $Z(\Theta)$ for the zero set $\{m \in \mathfrak{M}(H^\infty) : m(\Theta) = 0\}$ of Θ in the maximal ideal space $\mathfrak{M}(H^\infty)$ and $Z_{\mathbb{D}}(\Theta)$ for the zero set of Θ in the disc.

3. Norm controlled inversions in $H^\infty/\Theta H^\infty$ and the WEP

The main results of this section are that the norm controlled inversion property for the quotient algebra $H^\infty/\Theta H^\infty$ is equivalent to the WEP (Theorems 3.3 and 3.4) and that the WEP is not equivalent to the CEP (Examples 3.7 and 3.8). We also give an upper estimate of the function $c_n(H^\infty/\Theta H^\infty, \Lambda, \delta)$ in terms of the WEP characteristic η_Θ , where Λ is the zero set of Θ in \mathbb{D} .

We begin by recalling two known lemmas. The first one is a version of a classical result of Kerr-Lawson [9] and Hoffman [6]. For a proof, see [2, Lemma 1.4, p. 404].

Lemma 3.1 (Hoffman's lemma). Suppose b is an interpolating Blaschke product with zeros $\{z_n : n \in \mathbb{N}\}$ and let $\delta(b)$ be its Carleson separation constant. If $0 < \delta < \delta(b)$ and $0 < \epsilon < (1 - \sqrt{1 - \delta^2})/\delta$, then $\{z \in \mathbb{D} : |b(z)| < \epsilon^2\}$ is the union of pairwise disjoint domains V_n with $z_n \in V_n$ and $V_n \subset \{z : \rho(z, z_n) < \epsilon\}$.

The second lemma is also well known (see [10, p. 218]).

Lemma 3.2. Suppose that for $z, \lambda \in \mathbb{D}$ we have $\rho(z, \lambda) < \delta/3$. Then

$$|\lambda - z| \leq \delta/2 \min\{1 - |\lambda|^2, 1 - |z|^2\}.$$

In this section we prove the following theorem.

Theorem 3.3. Let Θ be an inner function on \mathbb{D} . The following are equivalent.

- (1) The quotient algebra $H^\infty/\Theta H^\infty$ has no corona; that is, $\Lambda = \sigma(\Theta) \cap \mathbb{D}$ is dense in $\mathfrak{M}(H^\infty/\Theta H^\infty)$.
- (2) If $f \in H^\infty$ and $\delta_1(f, \Lambda) > 0$, then $f + \Theta H^\infty$ is invertible in $H^\infty/\Theta H^\infty$.
- (3) $\delta_n(H^\infty/\Theta H^\infty, \Lambda) = 0$ for every $n \geq 1$.
- (4) $\delta_1(H^\infty/\Theta H^\infty, \Lambda) = 0$.

- (5) Θ satisfies the WEP.
 (6) $\Theta = BS_\mu$ where $\overline{Z_{\mathbb{D}}(B)} = Z(B) = Z(\Theta)$.

Moreover, if (1)–(6) hold, then

$$c_n(H^\infty/\Theta H^\infty, \Lambda, \delta) \leq 18\sqrt{n+1} \frac{\log(\frac{1}{\eta_\Theta(\delta/3)})}{[\eta_\Theta(\delta/3)]^2},$$

where $\eta_\Theta(\epsilon)$ is the WEP-characteristic of Θ .

There is an alternative way of looking at the theorem above. For instance, statement (2) says that $f + \Theta H^\infty \in H^\infty/\Theta H^\infty$ is invertible if and only if $\inf_{\lambda \in \Lambda} |f(\lambda)| > 0$.

Statement (3) means that for every $f = (f_1, \dots, f_n) \in (H^\infty)^n$ satisfying $0 < \delta_n(f, \Lambda) \leq \|f\| \leq 1$ there exist solutions $h \in H^\infty$ and $g = (g_1, \dots, g_n) \in (H^\infty)^n$ to the Bezout equation

$$\sum_{j=1}^n f_j g_j + \Theta h = 1$$

with a norm control in terms of $\delta_n(f, \Lambda)$ only:

$$\|g\| = \left(\sum_{j=1}^n \|g_j\|_A^2 \right)^{1/2} \leq c_n(\Lambda, \delta_n(f, \Lambda)).$$

The proof of Theorem 3.3 makes use of Carleson's corona theorem. The aforementioned estimate will follow from the known estimates in the corona theorem (the best known one is given by S. Treil and B. Wick in [19]).

The extra factor of $\sqrt{n+1}$ appears because, in place of the norm used in [10,19] or [2], which is

$$\|F\|_{H_{n+1}^\infty} = \sup_{z \in \mathbb{D}} |F(z)|, \quad \text{where } |F(z)| = \left(\sum_{j=1}^{n+1} |F_j(z)|^2 \right)^{1/2},$$

we work (following the general setting) with the norm

$$\|F\| = \left(\sum_{j=1}^{n+1} \|F_j\|_\infty^2 \right)^{1/2}$$

that arises from the corresponding expression in $H^\infty/\Theta H^\infty$ satisfying

$$\|F\| \leq \sqrt{n+1} \|F\|_{H_{n+1}^\infty}.$$

This choice is quite natural, because $H^\infty/\Theta H^\infty$ is not, in general, a uniform algebra and therefore $\|\cdot\|_{H_n^\infty}$ is not defined. Moreover, for other Banach algebras in which Bezout equations are studied (see [12,14]) the use of norms like $\|\cdot\|$ is common.

We will prove the following equivalent form of Theorem 3.3.

Theorem 3.4. Let Θ be an inner function on \mathbb{D} and let Λ be its zero set in \mathbb{D} . The following are equivalent.

- (1) The quotient algebra $H^\infty/\Theta H^\infty$ has no corona; that is, Λ is dense in $\mathfrak{M}(H^\infty/\Theta H^\infty)$.
- (2) Given $f \in H^\infty$ there exist $g, h \in H^\infty$ such that $fg + \Theta h = 1$ if and only if $\delta(f, \Lambda) := \inf_{\lambda \in \Lambda} |f(\lambda)| > 0$.
- (3) Given $f = (f_1, \dots, f_n) \in (H^\infty)^n$ with

$$\|f\| := \left(\sum_{j=1}^n \|f_j\|_\infty^2 \right)^{1/2} \leq 1$$

and

$$\delta_n(f, \Lambda) = \inf_{\lambda \in \Lambda} \left(\sum_{j=1}^n |f_j(\lambda)|^2 \right)^{1/2} > 0,$$

the Bezout equation

$$\sum_{j=1}^n f_j g_j + \Theta h = 1$$

has a solution $(g, h) := (g_1, \dots, g_n, h) \in (H^\infty)^{n+1}$ such that

$$\|g\| := \left(\sum_{j=1}^n \|g_j\|^2 \right)^{1/2} \leq c_n(H^\infty/\Theta H^\infty, \Lambda, \delta_n(f, \Lambda)).$$

- (4) For $f \in H^\infty$ with $0 < \delta(f, \Lambda) \leq \|f\| \leq 1$, there is a solution to the Bezout equation $fg + \Theta h = 1$ with $\|g\| \leq c_1(\delta(f, \Lambda))$.
- (5) Θ satisfies the WEP.
- (6) $\Theta = BS_\mu$ where $\overline{Z_{\mathbb{D}}(B)} = Z(B) = Z(\Theta)$; in particular $Z(S_\mu) \subseteq Z(B)$.

Moreover, if (1)–(6) hold, then, for any $\delta \in]0, 1[$,

$$c_n(H^\infty/\Theta H^\infty, \Lambda, \delta) \leq 18\sqrt{n+1} \frac{\log(\frac{1}{\eta_\Theta(\delta/3)})}{[\eta_\Theta(\delta/3)]^2},$$

where $\eta_\Theta(\epsilon)$ is the WEP-characteristic of Θ .

Proof. Since $\mathfrak{M}(H^\infty/\Theta H^\infty)$ can be identified with $Z(\Theta)$ (which is well known), it follows that (1) \Leftrightarrow (6). Moreover, the implications (3) \Rightarrow (4) \Rightarrow (2) as well as (3) \Rightarrow (1) \Rightarrow (2) are obvious. So it remains to prove (2) \Rightarrow (5) and (5) \Rightarrow (3).

First we show that (5) \Rightarrow (3). Let $f = (f_1, \dots, f_n) \in (H^\infty)^n$ and suppose $0 < \delta = \delta_n(f, \Lambda) = \inf_{\lambda \in \Lambda} |f(\lambda)| \leq \|f\| \leq 1$, where $|f(\lambda)| = (\sum_{j=1}^n |f_j(\lambda)|^2)^{1/2}$ and $\|f\| = (\sum_{j=1}^n \|f_j\|_\infty^2)^{1/2}$. Let $z \in \mathbb{D}$ satisfy $\inf_{\lambda \in \Lambda} |b_\lambda(z)| < \delta/3$ and let λ be such that $|b_\lambda(z)| \leq \delta/3$. As in [10, p. 218]

we note that for $h \in H^\infty$ if we define the conformal map $\tau(z) = (w - z)(1 - \bar{w}z)^{-1}$ and write $g = h \circ \tau^{-1}$, then

$$|h'(w)| = |g'(0)| |\tau'(w)| \leq \|g\|_\infty |\tau'(w)| = \|h\|_\infty (1 - |w|^2)^{-1},$$

where the inequality follows from Cauchy's formula. Therefore, for $\lambda \in \Lambda$,

$$\begin{aligned} |f(z)| &\geq |f(\lambda)| - |f(\lambda) - f(z)| \\ &\geq \delta - |\lambda - z| \cdot \left(\sum_j \max_{t \in [\lambda, z]} |f'_j(t)|^2 \right)^{1/2} \\ &\geq \delta - |\lambda - z| \cdot \sum_j \max_{t \in [\lambda, z]} \|f_j\|_\infty (1 - |t|^2)^{-1} \\ &= \delta - |\lambda - z| \cdot \sum_j \|f_j\| / \min(1 - |\lambda|^2, 1 - |z|^2) \\ &\geq \delta - |\lambda - z| / \min(1 - |\lambda|^2, 1 - |z|^2). \end{aligned}$$

If $|b_\lambda(z)| \leq \delta/3$, by Lemma 3.2

$$|\lambda - z| \leq \delta/2 \cdot \min\{(1 - |\lambda|^2), (1 - |z|^2)\}.$$

Consequently, for z satisfying $\inf_{\lambda \in \Lambda} |b_\lambda(z)| < \delta/3$ we have

$$|f(z)| \geq \delta/2.$$

On the other hand, if z satisfies $\inf_{\lambda \in \Lambda} |b_\lambda(z)| \geq \delta/3$, then (see property (P1), Section 2), $|\Theta(z)| \geq \eta_\Theta(\delta/3)$. Thus by property (P2), Section 2, $\eta_\Theta(\delta/3) \leq \delta/3 < 1/e$, we obtain

$$|f(z)|^2 + |\Theta(z)|^2 \geq \min\{(\delta/2)^2, \eta_\Theta(\delta/3)^2\} = \eta_\Theta(\delta/3)^2$$

for every $z \in \mathbb{D}$. By Carleson's corona theorem and the estimates of solutions of corona equations from [19], there exists a solution $(g, h) := (g_1, \dots, g_n, h)$ of the Bezout equation

$$\sum_{j=1}^n f_j g_j + \Theta h = 1$$

such that

$$\begin{aligned} \|g\| &\leq \sqrt{n+1} \sup_z \left(\sum_{j=1}^n |g_j(z)|^2 + |h(z)|^2 \right)^{1/2} \\ &\leq \sqrt{n+1} \left[\frac{1}{\eta_\Theta(\delta/3)} + \frac{17}{\eta_\Theta(\delta/3)^2} \log \frac{1}{\eta_\Theta(\delta/3)} \right] \\ &\leq \sqrt{n+1} \left(\frac{18}{\eta_\Theta(\delta/3)^2} \log \frac{1}{\eta_\Theta(\delta/3)} \right). \end{aligned}$$

This completes the proof of (5) \Rightarrow (3) and the estimate of inverses claimed in Theorems 3.3 and 3.4.

We turn to the final implication showing that (2) \Rightarrow (5).

First assume that Θ is a Blaschke product; that is, $\Theta = B = \prod_j b_{\lambda_j}^{k_j}$, where (λ_j) denotes the zero sequence of Θ .

Let ϵ be a positive number so that $\epsilon < 1$. Consider the sets

$$\Omega(\epsilon) := \Omega_\epsilon \left(\inf_j |b_{\lambda_j}| \right) = \bigcup_j \{z \in \mathbb{D}: |b_{\lambda_j}(z)| < \epsilon\},$$

and

$$\Omega_\epsilon(B) = \{z: |B(z)| < \epsilon\}.$$

Clearly, $\Omega(\epsilon) \subseteq \Omega_\epsilon(B)$. Note also that for every $r \in]0, 1[$ and every $\epsilon > 0$, there exists an $\eta > 0$ such that

$$\Omega_\eta(B) \cap \{z \in \mathbb{D}: |z| \leq r\} \subseteq \Omega(\epsilon);$$

for example, we may take $\eta = \epsilon^M \min_{|z| \leq r} \prod_{j=N+1}^\infty |b_{\lambda_j}^{k_j}(z)|$, where $M = \sum_{j=1}^N k_j$ and N is chosen so large that $|\lambda_j| > r$ for every $j \geq N$.

We want to prove that for every $\epsilon > 0$ there exists $\eta > 0$ such that

$$\Omega_\eta(B) \subseteq \Omega(\epsilon) \tag{3.1}$$

(in other words, $\{|B| < \eta\} \subseteq \bigcup_j D_\rho(\lambda_j, \epsilon)$). To this end, suppose to the contrary that there exists $\epsilon > 0$ such that for all $\eta > 0$ we have $\Omega_\eta(B) \setminus \Omega(\epsilon) \neq \emptyset$. Thus, if $z_1 \in \Omega_{1/2}(B) \setminus \Omega(\epsilon)$, then $|b_{\lambda_j}(z_1)| > \epsilon$ for every j and $|B(z_1)| \leq 1/2$.

Now let $r = 1 - (1 - |z_1|)/2$ and choose $n_2 > 2$ so that

$$\Omega_{1/n_2}(B) \cap \{z: |z| \leq r\} \subseteq \Omega(\epsilon).$$

Then

$$\Omega_{1/n_2}(B) \setminus \Omega(\epsilon) \subseteq \{z \in \mathbb{D}: 1 - |z| < (1 - |z_1|)/2\}.$$

Taking $z_2 \in \Omega_{1/n_2}(B) \setminus \Omega(\epsilon)$ we get $1 - |z_2| < (1 - |z_1|)/2$, $|b_{\lambda_j}(z_2)| > \epsilon$ for all j , and $|B(z_2)| \leq 1/n_2$. Continuing in this way, we obtain sequences (n_k) and (z_k) such that

- (a) $|b_{\lambda_j}(z_k)| = |b_{z_k}(\lambda_j)| > \epsilon$ for all j and all k ;
- (b) $|B(z_k)| \leq 1/n_k$ where $n_k > n_{k-1}$;
- (c) $1 - |z_k| < (1 - |z_{k-1}|)/2$.

Using [10, p. 159] and property (c), we see that (z_k) is an interpolating sequence. Let $B_1 = \prod_{k=1}^\infty b_{z_k}$ be the corresponding interpolating Blaschke product. Then [10, p. 218] implies that there exists a constant γ such that

$$|B_1(z)| \geq \gamma \inf_j |b_{z_j}(z)|. \tag{3.2}$$

Now property (a) implies that $|B_1(\lambda_j)| \geq \gamma\epsilon$ for every j . Property (b) implies that

$$\inf\{|B_1(z)|^2 + |B(z)|^2 : z \in \mathbb{D}\} \leq \inf_k |B(z_k)|^2 = 0.$$

Therefore, we cannot find H^∞ functions g and h such that $B_1g + Bh = 1$. This contradicts our hypothesis (2) of Theorem 3.4. Hence our assumption that $\Omega_\eta(B) \setminus \Omega(\epsilon) \neq \emptyset$ was wrong. Thus we have established (3.1) and so we proved statement (5) of Theorem 3.4 in the case when Θ is a Blaschke product.

Now suppose that $\Theta = BS_\mu$ is an arbitrary inner function satisfying (2). Note that since $\Theta H^\infty = BS_\mu H^\infty \subseteq BH^\infty$, it follows that for all f if $f + \Theta H^\infty$ is invertible in $H^\infty/\Theta H^\infty$, then $f + BH^\infty$ is invertible in H^∞/BH^∞ . Therefore, the Blaschke factor of Θ also satisfies property (2).

We turn now to the singular factor of Θ . First we show that for every $\alpha > 0$ there exists $\beta > 0$ such that $\{|S_\mu| < \beta\} \subseteq \{|B| < \alpha\}$. Suppose this is not the case. Then there exists $\alpha_0 > 0$ such that for every $\beta = 1/m$, where $m = 1, 2, \dots$, there is a point $z_m \in \mathbb{D}$ such that $|S(z_m)| < 1/m$, but $|B(z_m)| \geq \alpha_0$. Without loss of generality, we may assume that (z_m) is an interpolating sequence. Let b be the associated interpolating Blaschke product. Recall that (λ_k) denotes the zero sequence of B . Since $|b_{\lambda_k}(z_m)| \geq |B(z_m)| \geq \alpha_0$ for every m and k , we obtain (by Lemma 3.1 or formula (3.2)), that $|b(\lambda_k)|$ is bounded away from zero.

Thus assertion (2) implies that $b + \Theta H^\infty$ is invertible in $H^\infty/\Theta H^\infty$; that is there exist f and g in H^∞ such that $fb + g\Theta = 1$. But

$$\inf\{|b(z)|^2 + |\Theta(z)|^2 : z \in \mathbb{D}\} \leq \inf_m |S_\mu(z_m)|^2 = 0.$$

This is a contradiction. Thus, for every $\alpha > 0$ there exists $\beta > 0$ such that $\Omega_\beta(S_\mu) \subseteq \Omega_\alpha(B)$. Without loss of generality, we may assume that $\beta \leq \alpha$.

We have already shown that $B \in \text{WEP}$, so for $\epsilon > 0$ we may choose $\alpha > 0$ so that $\Omega_\alpha(B) \subseteq \Omega_\epsilon(\inf_\lambda |b_\lambda|)$. Let $\beta \leq \alpha$ be as above. We claim that by setting $\eta = \beta^2$, we obtain $\Omega_\eta(\Theta) \subseteq \Omega_\epsilon(\inf_\lambda |b_\lambda|)$; that is, we claim that assertion (5) of the theorem holds. In fact, if $|\Theta(z)| = |B(z)S_\mu(z)| < \beta^2$, then either $|B(z)| < \beta \leq \alpha$ and hence $z \in \Omega_\epsilon(\inf_\lambda |b_\lambda|)$, or $|S_\mu(z)| < \beta$ and then $|B(z)| < \alpha$, and again $z \in \Omega_\epsilon(\inf_\lambda |b_\lambda|)$. \square

We derive two immediate corollaries. First we formalize the “splitting property” of WEP inner functions that appeared at the end of the previous proof.

Corollary 3.5. *Let $\Theta = BS_\mu$ be an inner function. The following assertions are equivalent:*

- (a) $\Theta \in \text{WEP}$;
- (b) $B \in \text{WEP}$ and for every $\alpha > 0$ there exists $\beta > 0$ such that $\Omega_\beta(S_\mu) \subseteq \Omega_\alpha(B)$.

Proof. By the theorem above, (a) implies assertion (2) of Theorems 3.3, 3.4. Then the second to the last paragraph of the previous proof shows that (b) holds. The last paragraph of the same proof above shows that (b) implies (a). \square

Corollary 3.6. *Let N be an integer and B a product of N interpolating Blaschke products. Then assertions (1)–(6) in Theorem 3.4 hold true for B . Moreover, for every $\delta \in]0, 1[$ and $n \geq 1$ there exists a constant $a > 0$ (depending on B , and in particular on N), such that*

$$c_n(H^\infty/BH^\infty, \Lambda, \delta) \leq a\sqrt{n+1} \frac{\log(\frac{1}{\delta/3})}{\delta^{2N}}.$$

Proof. This follows from Theorem 3.4 and property (3) of Section 2. \square

In Section 2 we discussed Kerr-Lawson's lemma, which shows that if B is a finite product of interpolating Blaschke products with zero sequence (z_n) , then for every $\epsilon \in]0, 1[$ there exists $\eta > 0$ such that

$$\{z \in \mathbb{D}: |B(z)| < \eta\} \subseteq \bigcup_n D_\rho(z_n, \epsilon). \quad (3.3)$$

Thus, as we noted in Section 2 (property (P3)), every such Blaschke product satisfies the weak embedding property. Kerr-Lawson [9] also proved that if u is an inner function and for some $\eta > 0$ and $\epsilon \in]0, 1[$ the set $\{z \in \mathbb{D}: |u(z)| < \eta\}$ is contained in the *disjoint* union $\bigcup_n D_\rho(z_n, \epsilon)$ of pseudohyperbolic discs of fixed radius ϵ and centers z_n , then u is a finite product of interpolating Blaschke products. In the last paragraph of his paper he asserted that this implies, in particular, that every Blaschke product that is not a finite product of interpolating Blaschke products is arbitrarily small arbitrarily far away from its zeros inside \mathbb{D} .

The following example shows that this is not the case; indeed the class of WEP Blaschke products is strictly larger than the class of finite products of interpolating Blaschke products. As indicated in the introduction, the main idea for these examples is due to S. Treil and was realized in a correspondence between V. Vasyunin and the third author of this paper. We are indebted to Professors Treil and Vasyunin for permitting us to include the example in this paper.

Clearly, WEP, as defined in Section 2, is conformally invariant. In particular, we can replace the disc \mathbb{D} by the upper-half plane $\mathbb{C}_+ = \{z \in \mathbb{C}: \operatorname{Im} z > 0\}$ without changing the definitions. Since the pseudohyperbolic geometry of \mathbb{C}_+ is more transparent than that of \mathbb{D} , we give our principal example in \mathbb{C}_+ (Example 3.7 below).

Example 3.7. There exists a WEP Blaschke product B that does not satisfy the CEP (that is, B satisfies the WEP, but B is not a finite product of interpolating Blaschke products).

Proof. The construction is done using techniques from [10]. Let $a > 0$ and let $\Theta(z) = e^{iaz}$ be an inner function on $\mathbb{C}_+ = \{z: \operatorname{Im} z > 0\}$. For $t > 0$, consider the Frostman shift $B_{a,t}$ of Θ given by

$$B_{a,t}(z) = \frac{\Theta(z) - e^{-t}}{1 - e^{-t}\Theta(z)}.$$

Then $B_{a,t}$ is a Blaschke product with zeros $z_{k,a,t} = 2\pi k/a + it/a$, $k \in \mathbb{Z}$. It satisfies the Carleson interpolation condition and the Blaschke mass of its zeros is

$$\sum_k \frac{at}{t^2 + 4\pi^2 k^2} \sim a.$$

Now take $a = 1/n^2$ and $t = n$ for $n = 1, 2, \dots$. Define the Blaschke product B by

$$B = \prod_n B_{1/n^2, n}.$$

This will be the Blaschke product we seek. Note that B has the zeros $z_{k,n} = 2\pi kn^2 + in^3$, for $k \in \mathbb{Z}$ and $n \geq 1$. Moreover, we have the following:

- (1) For every $\epsilon > 0$ the level set

$$\Omega_\epsilon = \bigcup_{k,n} \{z \in \mathbb{C}_+ : |z - z_{k,n}| < \epsilon n^3\}$$

contains a half-plane; for example $\Pi_\epsilon = \{z \in \mathbb{C}_+ : \operatorname{Im} z > (100/\epsilon)^3\}$. The zeros of B outside Π_ϵ form a Carleson interpolating sequence, hence $B^{(1)} := \prod_{n \leq 100/\epsilon} B_{1/n^2, n}$ is an interpolating Blaschke product.

- (2) A direct look at

$$\prod \left| \frac{e^{iz/n^2} - e^{-n}}{1 - e^{-n} e^{iz/n^2}} \right|$$

shows that the subproduct $B^{(2)} := \prod_{n > 100/\epsilon} B_{1/n^2, n}$ is bounded away from zero on the strip $S_\epsilon = \{z \in \mathbb{C}_+ : 0 < \operatorname{Im} z \leq (100/\epsilon)^3\}$.

- (3) Since (1) and (2) are valid for every $\epsilon > 0$ and $B = B^{(1)} B^{(2)}$, for sufficiently small $\eta > 0$, the part of the level set $S_\epsilon \cap \{z \in \mathbb{C}_+ : |B(z)| < \eta\}$ is included in the union

$$\bigcup_{k \in \mathbb{Z}, n < 100/\epsilon} \{z \in \mathbb{C}_+ : |z - z_{k,n}| < \epsilon n^3\},$$

and hence $\{z \in \mathbb{C}_+ : |B(z)| < \eta\} \subseteq \Omega_\epsilon$.

- (4) The three statements above show that for every $\epsilon > 0$ there exists $\eta > 0$ such that $\{z \in \mathbb{C}_+ : |B(z)| < \eta\} \subseteq \Omega_\epsilon$, but B is not a finite product of interpolating Blaschke products (because, for example, the Carleson embedding theorem is not satisfied: $\max \sum_{x+iy \in Q} y$, where the summation runs over all zeros $x + iy$ of B in the square $Q = \{0 \leq x, y \leq a\}$, is at least of the order $a^{4/3}$ as $a \rightarrow \infty$, and not of the order of a as required for Carleson measures). \square

It is of interest to know whether there are inner functions satisfying the WEP that are not Blaschke products. The next example shows that such inner functions exist.

Example 3.8. Let B be the Blaschke product of Example 3.7 and let $S(z) = e^{iaz}$, where $a > 0$. Then the function $\Theta = SB$ is a WEP inner function.

Proof. Indeed, keeping the notation of the previous example, we will prove that for every $\epsilon > 0$, there exists $\eta > 0$ such that $\Omega_\eta(\Theta) \subset \Omega_\epsilon$. Let Π_ϵ be the half-plane defined in Example 3.7 above. For $z \in \mathbb{C}_+ \setminus \Pi_\epsilon$, we have

$$|\Theta(z)| \geq e^{-a} |B(z)|,$$

where $\alpha = a(100/\epsilon)^3$. Hence, by Example 3.7,

$$\Omega(\Theta) \cap (\mathbb{C}_+ \setminus \Pi_\epsilon) \subseteq \Omega_{\eta e^\alpha}(B) \subset \Omega_\epsilon$$

for $\eta > 0$ sufficiently small. As we saw in Example 3.7 above, $\Pi_\epsilon \subset \Omega_\epsilon$, so the inclusion $\Omega_\eta(\Theta) \subset \Omega_\epsilon$ follows. \square

At this point, we mention a few additional properties of WEP Blaschke products and WEP sequences. Properties (P1)–(P4) can be found in Section 2.

(P5) *A separated WEP sequence is interpolating.*

Proof. Let $\Lambda = (\lambda_j)$ be a separated WEP-sequence and let B be the associated Blaschke product. Then, for every $\epsilon \in]0, 1[$ there exists $\eta = \eta(\epsilon) > 0$ such that

$$\{|B| < \eta\} \subseteq \bigcup_{\lambda \in \Lambda} D_\rho(\lambda, \epsilon).$$

Since Λ is separated, there exists $\epsilon' > 0$ such that the discs $D = D_\rho(\lambda, \epsilon')$, $\lambda \in \Lambda$, are pairwise disjoint. On the boundary, ∂D , of such a disc we have

$$|B(z)| \geq \eta(\epsilon') = \frac{\eta(\epsilon')}{\epsilon'} |b_\lambda(z)|.$$

By the minimum modulus principle, the function $\varphi: z \mapsto B(z)/b_\lambda(z)$, which is holomorphic and zero free on \overline{D} , satisfies the same estimate $|\varphi(z)| \geq \eta(\epsilon')/\epsilon'$ on this disc. Taking $z = \lambda$, we get the Carleson interpolation condition $\prod_{j \neq k} |b_{\lambda_j}(\lambda_k)| \geq \eta(\epsilon')/\epsilon' > 0$ for every $\lambda_k \in \Lambda$. \square

(P6) *Let Λ be a finite union of separated sequences and suppose that $\Lambda \in \text{WEP}$. Then $\Lambda \in \text{CEP}$.*

Proof. Assume that Λ is a union of N separated sequences and let $B = \prod_{\lambda \in \Lambda} b_\lambda$. Consider the open set $\Omega_\epsilon(\inf_{\lambda \in \Lambda} |b_\lambda|) = \bigcup_{\lambda \in \Lambda} D_\rho(\lambda, \epsilon)$ and its open connected components $\Omega^1, \Omega^2, \dots$. The triangle inequality implies that for $\epsilon > 0$ small enough every Ω^j contains no more than N points of Λ . Fixing such an $\epsilon > 0$, we apply WEP, which gives

$$\Omega_\eta(B) \subseteq \Omega_\epsilon\left(\inf_{\lambda \in \Lambda} |b_\lambda|\right) = \bigcup_j \Omega^j$$

for some $\eta > 0$. Then every connected component of $\Omega_\eta(B)$ is contained in one of Ω^j , hence contains no more than N points of Λ . But it is known that a Blaschke product is the product of at most N interpolating Blaschke products if and only if there exists $\eta > 0$ such that every connected component of the level set $\{z: |B(z)| < \eta\}$ contains at most N zeros of B . This fact is implicitly contained in [11, pp. 229, 230]. It can also be found in [13, 15] and [14, vol. 2, p. 189]. Thus $\Lambda \in \text{WEP}$. \square

(P7) Let $\zeta \in \partial \mathbb{D}$ and let $0 < \theta < \pi/2$. Let Λ be a sequence in a Stolz angle $S(\zeta, \theta)$ (that is the convex hull of the disk $|z| \leq \sin \theta$ and the point ζ). Then $\Lambda \in \text{WEP}$ if and only if $\Lambda \in \text{CEP}$.

Proof. By definition, we see that a function satisfying CEP also satisfies WEP. To prove the converse, since a separated sequence in a Stolz angle is interpolating, it suffices to show that Λ is a finite union of separated sequences. It is also clear that for the latter property it is sufficient to prove that there exist β and N , $0 < \beta < 1$, such that every disc $D_\rho(\lambda, \beta)$, $\lambda \in \Lambda$, contains no more than N points of Λ (the standard dyadic “white-and-black-boxes” reasoning gives the proof, see for instance [2], or [10, p. 159]). In order to ensure the last property, observe that if $\theta < \theta' < \pi/2$, then

$$\alpha := \inf\{|b_z(\lambda)| : z \in \partial S(\zeta, \theta'), \lambda \in \Lambda\} > 0.$$

On the other hand, there exists $\epsilon \in]0, 1[$, close to 1, such that $D_\rho(w, \epsilon) \cap \partial S(\zeta, \theta') \neq \emptyset$ for every $w \in S(\zeta, \theta)$. Hence, for every $\lambda \in \Lambda$, there exists $z = z(\lambda) \in \partial S(\zeta, \theta')$ such that $|b_z(\lambda)| < \epsilon$. Therefore, if $\beta > 0$ and $\gamma := \epsilon + \beta < 1$, then we have $D_\rho(\lambda, \beta) \subseteq D_\rho(z(\lambda), \gamma)$. Now, denoting $N(\lambda) = \text{card } D_\rho(\lambda, \beta) \cap \Lambda$, we get

$$|B(z(\lambda))| \leq \prod_{\mu} |b_\mu(z(\lambda))| \leq \gamma^{N(\lambda)},$$

where the product is taken over all $\mu \in D_\rho(\lambda, \beta) \cap \Lambda$. Since the WEP implies that $|B(z(\lambda))| \geq \eta_B(\alpha)$, where B is the Blaschke product associated with Λ (see (P1), Section 2), we get $\sup_{\lambda \in \Lambda} N(\lambda) < \infty$. \square

Alternatively, by using maximal ideal space techniques, we can prove the statement in (P7) as follows. Let Θ satisfy the WEP and assume that the zero set Λ of Θ is contained in a Stolz-angle. By Theorem 3.3, (1) \Leftrightarrow (6), we have that $\overline{Z_{\mathbb{D}}(\Theta)} = Z(\Theta)$. By [6, Theorem 6.4] every point $m \in \mathfrak{M}(H^\infty)$ in the closure of a Stolz-angle belongs to the closure of an interpolating sequence. Thus, by [4], Θ is a finite product of interpolating Blaschke products and so $\Lambda \in \text{CEP}$. \square

(P8) *A finite product of WEP Blaschke products has the WEP property.*

Proof. Let B_j be the corresponding Blaschke products with zero sequences Λ_j , $j = 1, \dots, n$. By the WEP for B_j , there exists for every $\epsilon > 0$ some δ_j such that

$$\Omega_{\delta_j}(B_j) \subseteq \Omega_\epsilon\left(\inf_{\lambda \in \Lambda_j} |b_\lambda|\right).$$

Now, if $B = \prod_{j=1}^n B_j$, $\Lambda = \bigcup_{j=1}^n \Lambda_j$, and $\delta = \prod_{j=1}^n \delta_j$, then

$$\Omega_\delta(B) \subseteq \bigcup_{j=1}^n \Omega_{\delta_j}(B_j) \subseteq \bigcup_{j=1}^n \Omega_\epsilon\left(\inf_{\lambda \in \Lambda_j} |b_\lambda|\right) \subseteq \Omega_\epsilon\left(\inf_{\lambda \in \Lambda} |b_\lambda|\right).$$

Thus B is a WEP-inner function. \square

Since the definition of a WEP sequence does not involve the multiplicities of the zeros of the associated Blaschke product B , one can ask whether there exist WEP sequences whose multiplicities are unbounded.

(P9) *There exist WEP sequences whose point multiplicities are not bounded.*

Proof. In fact, using the notation of the Example 3.7, for a slowly growing sequence $p_n \rightarrow \infty$ (for example, $p_n = \lfloor \sqrt{n} \rfloor$), the product $B = \prod_{n \geq 1} B_{1/n^2, n}^{p_n}$ satisfies the WEP property. This can be checked in the same manner as Example 3.7. \square

(P10) *Let $\Lambda = (\lambda_j)$ be a WEP sequence with associated Blaschke product B and for $\epsilon > 0$ let $\Lambda(\epsilon)$ its subsequence of ϵ -isolated points; that is,*

$$\Lambda(\epsilon) = \left\{ \lambda_j \in \Lambda : \inf_{\lambda_k \neq \lambda_j} |b_{\lambda_j}(\lambda_k)| \geq \epsilon \right\}.$$

Then, the multiplicities N_j of points λ_j in $\Lambda(\epsilon)$ are uniformly bounded, namely,

$$N_j \leq \frac{\log \eta_{\Lambda}(\epsilon/2)}{\log(\epsilon/2)}.$$

Proof. To see this, let $\lambda_j \in \Lambda(\epsilon)$ and $z \in \mathbb{D}$ such that $|b_{\lambda_j}(z)| = \epsilon/2$. Then $|b_{\lambda}(z)| \geq \epsilon/2$ for every $\lambda \in \Lambda$. By the WEP-property (property (P1) of Section 2), $|B(z)| \geq \eta_B(\epsilon/2)$. On the other hand, $(\epsilon/2)^{N_j} = |b_{\lambda_j}(z)|^{N_j} \geq |B(z)|$. The result now follows. \square

Finally we remark that by [3], if Θ is WEP-inner function, then for every $a \in \mathbb{D} \setminus \{0\}$ with $|a|$ sufficiently small, the Frostman transform $(\Theta - a)/(1 - \bar{a}\Theta)$ is in CEP. In particular, Θ can be uniformly approximated by interpolating Blaschke products.

4. The quotient algebra $A(\mathbb{D})/\Theta H^\infty$

We note that it is still possible to make sense of Theorem 3.3 when we replace $\Lambda = \sigma(\Theta) \cap \mathbb{D}$ with $\Lambda = \sigma(\Theta)$. Rather than looking at H^∞ functions, we consider functions from the disc algebra, denoted here by $A(\mathbb{D})$. Recall that $A(\mathbb{D})/\Theta H^\infty$ is the canonical image of $A(\mathbb{D})$ in the quotient algebra $H^\infty/\Theta H^\infty$.

As mentioned in the introduction, the algebra $A(\mathbb{D})/\Theta H^\infty$ is closed in $H^\infty/\Theta H^\infty$ if and only if either $m(\sigma(\Theta) \cap \mathbb{T}) = 0$ or $m(\sigma(\Theta) \cap \mathbb{T}) = 1$. As we will see later on (Theorem 4.2), the latter case is not of interest for the efficient inversion problem. The former one, to the contrary, is very interesting. In this case, for the problem of norm controlled inversions, the algebra $A(\mathbb{D})/\Theta H^\infty$ is even more significant than $H^\infty/\Theta H^\infty$. The reason is that for $H^\infty/\Theta H^\infty$, the norm controlled inversion property (incidentally) coincides with the corona property, and hence the metric problem on the critical constants δ_n is, in a sense, hidden behind the topological fact that the visible spectrum Λ is dense in $\mathfrak{M}(H^\infty/\Theta H^\infty)$. For $A(\mathbb{D})/\Theta H^\infty$, these two properties are distinct: the algebra $A(\mathbb{D})/\Theta H^\infty$ never has a corona with respect to $\sigma(\Theta)$, but it may or may not have the norm-controlled inversion property. This phenomenon, which does appear in different situations (see [2,14,16]), is a specific internal property of a Banach algebra measuring the discrepancy between the Gelfand transform norm and the original algebra norm.

Lemma 4.1. *Let Θ be an inner function. The natural restriction embedding $f + \Theta H^\infty \mapsto f|_{\sigma(\Theta)}$ is a (contractive) homomorphism from the quotient algebra $A(\mathbb{D})/\Theta H^\infty$ into $C(\sigma(\Theta))$.*

If $m(\sigma(\Theta) \cap \mathbb{T}) = 0$, the maximal ideal space $\mathfrak{M}(A(\mathbb{D})/\Theta H^\infty)$ coincides with $\sigma(\Theta)$ with respect to this mapping. Consequently, given $f_j \in A(\mathbb{D})$ the Bezout equation $\sum_{j=1}^n g_j f_j = 1$ is

solvable in $A(\mathbb{D})/\Theta H^\infty$ if and only if $\sum_{j=1}^n |f_j(\lambda)|^2 > 0$ for every $\lambda \in \sigma(\Theta)$ (or if and only if $\inf_{\lambda \in \Lambda} (\sum_{j=1}^n |f_j(\lambda)|^2) > 0$, in the case in which Θ is a Blaschke product B and $\Lambda = B^{-1}(\{0\})$).

Proof. It is known that $\sigma(\Theta) = \{\lambda \in \bar{\mathbb{D}}: \liminf_{\zeta \rightarrow \lambda} |\Theta(\zeta)| = 0\}$ (see [13, p. 63]). Hence, for every $\lambda \in \sigma(\Theta)$, $f \in A(\mathbb{D})$, and $g \in H^\infty$, we have

$$|f(\lambda)| = \lim_{\zeta \rightarrow \lambda} |f(\zeta)| \leq \limsup_{\zeta \rightarrow \lambda} |f(\zeta) + \Theta(\zeta)g(\zeta)| \leq \|f + \Theta g\|_\infty,$$

and therefore, $\|f|_{\sigma(\Theta)}\|_{C(\sigma(\Theta))} \leq \|f\|_{H^\infty/\Theta H^\infty}$.

Now, assume that $m(\sigma(\Theta) \cap \mathbb{T}) = 0$. If φ is a complex continuous homomorphism on $A(\mathbb{D})/\Theta H^\infty$ and $\lambda = \varphi(z + \Theta H^\infty)$ (where $z(\zeta) \equiv \zeta$), then $\lambda \in \sigma(\Theta)$.

Indeed, $\lambda \in \bar{\mathbb{D}}$ since $\|\varphi\| \leq 1$, and hence φ extends naturally to a homomorphism of $A(\mathbb{D})$, $f \mapsto f(\lambda)$. Moreover, there exists an outer function g in $A(\mathbb{D})$ such that the zero set of g is $\sigma(\Theta) \cap \mathbb{T}$, which gives $0 = \varphi(\Theta g) = (\Theta g)(\lambda)$; that is, $\lambda \in \sigma(\Theta)$. Since the polynomials are dense in $A(\mathbb{D})$, we get $\varphi(f + \Theta H^\infty) = f(\lambda)$ for every $f \in A(\mathbb{D})$. Therefore, $\mathfrak{M}(A(\mathbb{D})/\Theta H^\infty) \subseteq \sigma(\Theta)$. Note that $A(\mathbb{D})$ distinguishes the points of $\sigma(\Theta)$. The reverse inclusion is obvious. \square

Theorem 4.2. *Let Θ be an inner function on \mathbb{D} . The following are equivalent.*

- (1) $\delta_n(A(\mathbb{D})/\Theta H^\infty, \sigma(\Theta)) = 0$ for every $n \geq 1$.
- (2) $\delta_1(A(\mathbb{D})/\Theta H^\infty, \sigma(\Theta)) = 0$.
- (3) $m(\sigma(\Theta) \cap \mathbb{T}) = 0$ and $\Theta \in \text{WEP}$.

Moreover, if (1)–(3) hold, then for every $\delta \in]0, 1[$,

$$c_n(A(\mathbb{D})/\Theta H^\infty, \Lambda, \delta) \leq 18\sqrt{n+1} \frac{\log(\frac{1}{\eta_\Theta(\delta/3)})}{[\eta_\Theta(\delta/3)]^2},$$

where, as usual, $\Lambda = \sigma(\Theta) \cap \mathbb{D}$ is the zero set of Θ and $\eta_\Theta(\epsilon)$ is the WEP-characteristic of Θ .

Before proceeding to the proof, we note that the essential part of Theorem 4.2 will follow from estimates of solutions of the Bezout equation in the disc algebra $A(\mathbb{D})$ as well as a generalization of the Rudin–Carleson theorem on free $A(\mathbb{D})$ -interpolation. By the latter, we mean the following: let $E \subset \bar{\mathbb{D}}$ be a closed set. Then $A(\mathbb{D})|_E = C(E)$ if and only if $m(E \cap \mathbb{D}) = 0$ and $E \cap \mathbb{D}$ is an interpolating sequence (see [5]). For our proof we will need the following lemma suggested by Sergei Treil. This lemma can also be found in [16].

Lemma 4.3. *Let $0 < \delta \leq 1$. Then $c_n(H^\infty, \mathbb{D}, \delta) = c_n(A(\mathbb{D}), \mathbb{D}, \delta)$.*

Proof. The inequality $c_n(H^\infty, \mathbb{D}, \delta) \leq c_n(A(\mathbb{D}), \mathbb{D}, \delta)$ is well known (see, for example, [10, Appendix 3]).

For the reverse inequality, let $n \geq 1$ and let $f = (f_1, \dots, f_n) \in (A(\mathbb{D}))^n$ satisfy $\delta_n(f, \mathbb{D}) = \inf_{\lambda \in \mathbb{D}} |f(\lambda)| > 0$. Let $\phi = (\phi_1, \dots, \phi_n) \in (H^\infty)^n$ be an H^∞ solution of the equation $\sum_{j=1}^n \phi_j f_j = 1$.

Now, given $\epsilon > 0$ there exists r with $0 < r < 1$ such that

$$\left| \sum_{j=1}^n \phi_j(rz) f_j(z) - 1 \right| = \left| \sum_{j=1}^n \phi_j(rz) (f_j(z) - f_j(rz)) \right| < \epsilon,$$

for every $z \in \mathbb{D}$. Therefore, the functions

$$g_j(z) = \frac{\phi_j(rz)}{\sum_{j=1}^n \phi_j(rz) f_j(z)}, \quad z \in \mathbb{D},$$

give a solution in $A(\mathbb{D})$ satisfying $\sum_{j=1}^n g_j f_j = 1$ with norm arbitrarily close to that of the solution given by ϕ_j . The inequality follows. \square

Proof of Theorem 4.2. It is clear that $(1) \Rightarrow (2)$.

We show that $(2) \Rightarrow (3)$. Note first that if $F \in A(\mathbb{D})$, then F is continuous at every point of the unit circle. If the inner factor of F is discontinuous at a point λ , its cluster set at the point λ is the closed unit disc [2, p. 80]. Therefore, F must vanish at the point λ . Now suppose $f, g \in A(\mathbb{D})$ and $fg + \Theta h = 1$. Letting Θh play the role of F above we conclude that $fg = 1$ on $\sigma(\Theta)$. If $m(\sigma(\Theta) \cap \mathbb{T}) > 0$, then $fg \equiv 1$. Since this must hold for every f that is bounded away from zero on $\sigma(\Theta)$ (and, in particular, for those f having a zero outside $\sigma(\Theta)$) we see that the condition $m(\sigma(\Theta) \cap \mathbb{T}) = 0$ is necessary for (2).

Now let $\Lambda = \sigma(\Theta) \cap \mathbb{D}$ and let $f \in H^\infty$ be such that

$$0 < \delta = \delta(f, \Lambda) = \inf_{\lambda \in \Lambda} |f(\lambda)| \leq \|f\|_\infty < 1.$$

By R. Nevanlinna's theorem (see [10, p. 204] or [13, vol. 1, p. 234]) there exists an inner function φ such that $\varphi|_\Lambda = f|_\Lambda$. Moreover, the same is true for any Blaschke set $\Lambda' \supset \Lambda$. By using this fact for $(1 + \frac{1}{n})f$ instead of f and for $\Lambda'_n = \Lambda \cup \{\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\}$ instead of Λ , and uniformly approximating the corresponding inner function φ by Blaschke products, we obtain a sequence of Blaschke products B_n such that

$$\sup_{z \in \Lambda'_n} \left| \left(1 + \frac{1}{n}\right) f(z) - B_n(z) \right| \leq \delta/n.$$

In particular $|B_n| \geq (1 + \frac{1}{n})|f| - |B_n - (1 + \frac{1}{n})f| \geq \delta$ on Λ .

Choosing convenient finite Blaschke subproducts C_n of B_n , we get a sequence (C_n) of functions in $A(\mathbb{D})$ such that $|C_n| \geq \delta$ on Λ and such that (C_n) converges locally uniformly on \mathbb{D} to f .

Now $C_j \in A(\mathbb{D})$ and $0 < \delta = \delta(f, \Lambda) \leq |C_j(\lambda)| \leq \|C_j\|_\infty = 1$ for every $\lambda \in \sigma(\Theta)$. By (2), there exist $g_n \in A(\mathbb{D})$ and $h_n \in H^\infty$ satisfying

$$C_n g_n + \Theta h_n = 1 \quad \text{and} \quad \|g_n\|_\infty \leq c_1(A(\mathbb{D})/\Theta H^\infty, \delta).$$

Using Montel's theorem we obtain functions g and h in H^∞ with

$$\|g\| \leq c_1(A(\mathbb{D})/\Theta H^\infty, \delta) \quad \text{and} \quad fg + \Theta h = 1.$$

In particular, statement (2) of Theorem 3.4 holds. Thus, it follows from Theorem 3.4 (statement (5)) that Θ has the desired property.

The last assertion of the theorem follows from the corresponding statement in Theorem 3.3 and Lemma 4.3.

Next we show (3) \Rightarrow (1). Let $\Lambda = \sigma(\Theta) \cap \mathbb{D}$ and $f = (f_1, \dots, f_n) \in (A(\mathbb{D}))^n$ be such that

$$0 < \delta = \delta_n(f, \Lambda) = \min_{\lambda \in \Lambda} |f(\lambda)| \leq \|f\| \leq 1.$$

The proof now proceeds in the same manner as that of Theorem 3.4. This yields the following estimation:

$$|f(z)|^2 + |\Theta(z)|^2 \geq \eta_\Theta(\delta/3)^2$$

for every $z \in \mathbb{D}$. Since f is continuous on $\overline{\mathbb{D}}$ and $\sigma(\Theta) \subseteq \overline{\Lambda}$, we see that $|f| \geq \eta_\Theta(\delta/3)$ on $\sigma(\Theta)$. Also, since $m(\sigma(\Theta) \cap \mathbb{D}) = 0$, there exists a peak function $p \in A(\mathbb{D})$ such that $p = 1$ on $\sigma(\Theta) \cap \mathbb{T}$ and $|p(z)| < 1$ for $z \in \overline{\mathbb{D}} \setminus (\sigma(\Theta) \cap \mathbb{T})$ [7, p. 80].

Now given $\epsilon > 0$, we may choose n sufficiently large, so that the function $\phi = 1 - p^n$ satisfies

$$|f(z)|^2 + |\Theta(z)\phi(z)|^2 \geq (1 - \epsilon)\eta_\Theta(\delta/3)^2$$

for every $z \in \mathbb{D}$. Now all the data, f_j and $\Theta\phi$, are in $A(\mathbb{D})$, so we may use Lemma 4.3 to conclude that there exists $g_1, \dots, g_{n+1} \in A(\mathbb{D})$ such that

$$\sum_{j=1}^n f_j g_j + \Theta\phi g_{n+1} = 1$$

and the norm $\|g\|$ of $g = (g_1, \dots, g_{n+1})$ is arbitrarily close to the norm of the best possible H^∞ solution. This shows that the lower bound for $A(\mathbb{D})$ -solutions is the same as for the best H^∞ -solutions. \square

5. Open questions

We conclude this paper with several open questions.

- (1) Find a geometric description of WEP sequences, introducing a “weak Carleson density” in place of the classical one that gives a description of the CEP sequences: $\mu(Q_h) \leq ch$, where $\mu = \sum_{\kappa(\lambda) > 0} (1 - |\lambda|^2) \delta_\lambda$ and where Q_h is a Carleson square with side h (see [2] or [10] for information on Carleson measures).
- (2) Describe possible singular factors S_μ of WEP inner functions $\Theta = S_\mu B$. We remark that it follows from Example 3.8 that a singular inner function S_μ with finite support, $\text{supp}(\mu)$, is admissible.
- (3) Is it true that finite products of interpolating Blaschke products are characterized by property (P3) of Section 2? It is sufficient to prove that if Λ is not in WEP, then $c_1(\Lambda, \delta)$ grows faster than any power of $1/\delta$ as $\delta \rightarrow 0$.

- (4) Let Θ be a non-WEP inner function, $\Theta = S_\mu B_A$. Is it true that

$$\delta_1(H^\infty/\Theta H^\infty, \Lambda) = 1 \quad \text{or} \quad \delta_1(A(\mathbb{D})/\Theta H^\infty, \sigma(\Theta)) = 1?$$

The latter statement makes sense even when $\Lambda = \emptyset$ (in which case the question is about an estimate of $\|f^{-1}\|_{A(\mathbb{D})/\Theta H^\infty}$ for functions $f \in A(\mathbb{D})$ satisfying

$$0 < \delta \leq |f(z)| \leq \|f\|_\infty \leq 1$$

for all $z \in \sigma(\Theta)$); in this case the answer is known to be “yes,” see [11].

- (5) Can every inner function be multiplied into the class WEP by a WEP-Blaschke product? This would solve, in particular, the long-standing open problem whether every point outside the Shilov boundary and having a trivial Gleason part lies in the closure of a Blaschke sequence.

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References

- [1] O. El-Fallah, N.K. Nikolski, M. Zarrabi, Estimates for resolvents in Beurling–Sobolev algebras, *Algebra i Analiz* 10 (6) (1998) 1–92 (in Russian, Russian summary); translated in: *St. Petersburg Math. J.* 10 (6) (1999) 901–964.
- [2] J.B. Garnett, *Bounded Analytic Functions*, Pure Appl. Math., vol. 96, Academic Press, New York, 1981, xvi+467.
- [3] P. Gorkin, R. Mortini, Two new characterizations of Carleson–Newman Blaschke products, preprint, 2007.
- [4] C. Guillory, K. Izuchi, D. Sarason, Interpolating Blaschke products and division in Douglas algebras, *Proc. R. Ir. Acad.* 84 (1984) 1–7.
- [5] E. Heard, J. Wells, An interpolation problem for subalgebras of H^∞ , *Pacific J. Math.* 28 (1969) 543–553.
- [6] K. Hoffman, Bounded analytic functions and Gleason parts, *Ann. of Math.* (2) 86 (1967) 74–111.
- [7] K. Hoffman, *Banach Spaces of Analytic Functions*, reprint of the 1962 original. Dover, New York, 1988.
- [8] K. Izuchi, Y. Izuchi, Algebras of bounded analytic functions containing the disk algebra, *Canad. J. Math.* 38 (1986) 87–108.
- [9] A. Kerr-Lawson, Some lemmas on interpolating Blaschke products and a correction, *Canad. J. Math.* 21 (1969) 531–534.
- [10] N.K. Nikolski, *Treatise on the Shift Operator. Spectral Function Theory*, with an appendix by S.V. Hruščev [S.V. Khrushchev] and V.V. Peller; translated from the Russian by J. Peetre, *Grundlehren Math. Wiss.*, vol. 273, Springer-Verlag, Berlin, 1986.
- [11] N.K. Nikolski, In search of the invisible spectrum, *Ann. Inst. Fourier (Grenoble)* 49 (6) (1999) 1925–1998.
- [12] N.K. Nikolski, The problem of efficient inversions and Bezout equations, in: *Twentieth Century Harmonic Analysis—A Celebration*, Il Ciocco, 2000, in: *NATO Sci. Ser. II Math. Phys. Chem.*, vol. 33, Kluwer Acad. Publ., Dordrecht, 2001, pp. 235–269.
- [13] N.K. Nikolski, *Operators, Functions, and Systems: An Easy Reading*, vols. 1, 2. *Model Operators and Systems*, *Math. Surveys Monogr.*, vol. 93, Amer. Math. Soc., Providence, RI, 2002.
- [14] N.K. Nikolski, Condition numbers of large matrices, and analytic capacities, *Algebra i Analiz* 17 (4) (2005) 125–180 (in English, English summary).

- [15] N.K. Nikolski, S.V. Khrushchëv, A functional model and some problems of the spectral theory of functions, *Tr. Mat. Inst. Steklov.* 176 (1987) 97–210, 327 (in Russian); translated in: *Proc. Steklov Inst. Math.* 3 (1988) 101–214.
- [16] S. Sidney, Sunwook Hwang, Sequence spaces of continuous functions, *Rocky Mountain J. Math.* 31 (2001) 641–659.
- [17] D. Stegenga, Sums of invariant subspaces, *Pacific J. Math.* 70 (1977) 567–584.
- [18] B. Szökefalvi-Nagy, C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, New York, 1970.
- [19] S. Treil, B. Wick, The matrix-valued H^p corona problem in the disk and polydisk, *J. Funct. Anal.* 226 (2005) 138–172.